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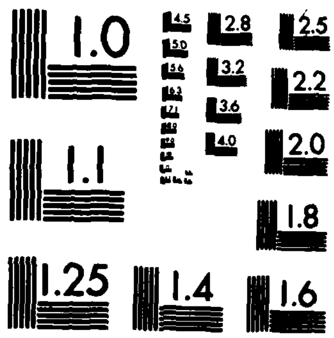
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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20084

AN INTRODUCTION TO THE FEARS PROGRAM

by

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Contract N00167-82M-0636

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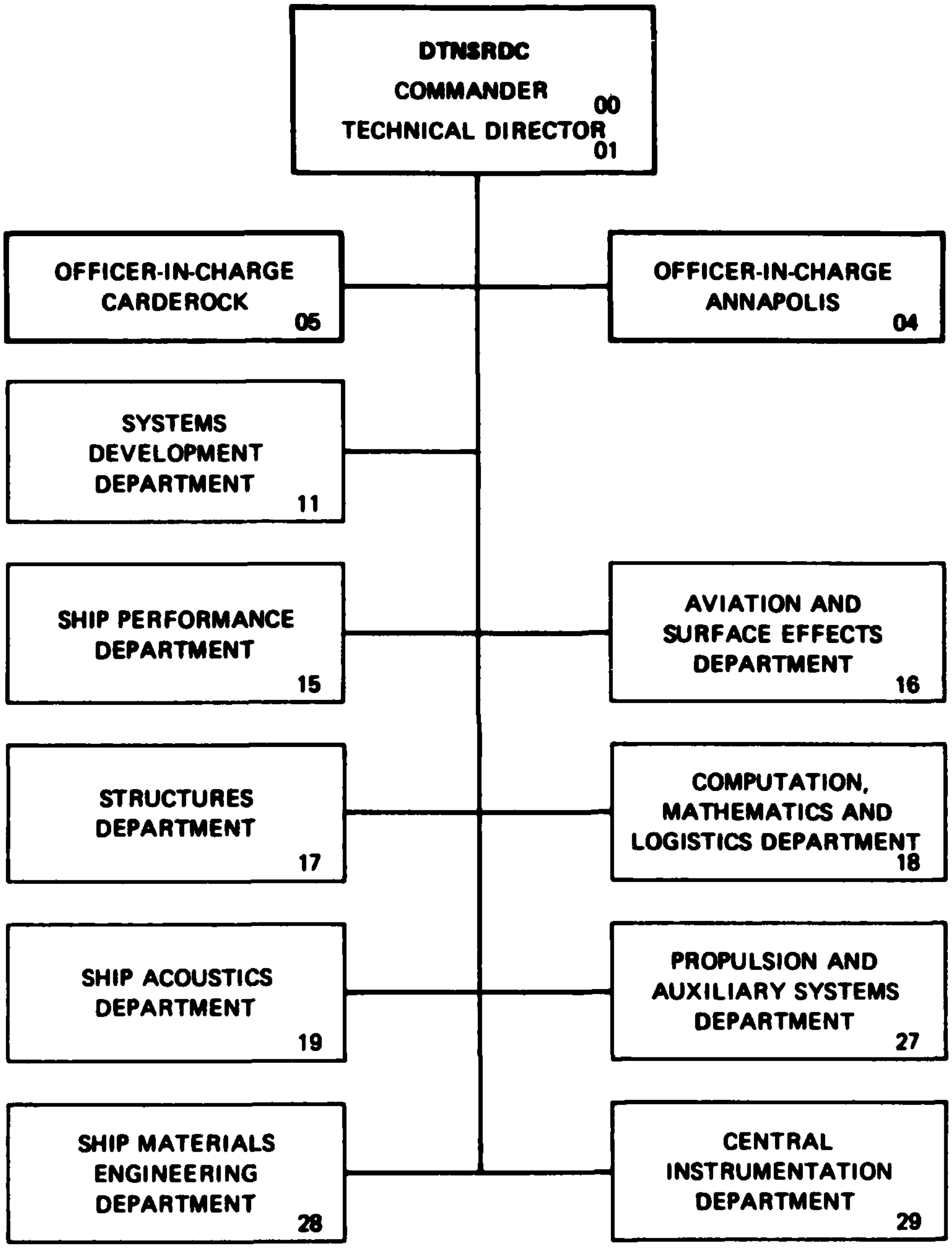

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This report is an introduction to the Finite Element Adaptive Research Solver (FEARS) computer program. It briefly describes the mathematical problem solved by FEARS. It tells how to prepare the input data for FEARS and how to interpret the output provided by FEARS. A composite shaft problem is provided as an example.

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ABSTRACT

This report is an introduction to the Finite Element Adaptive Research Solver (FEARS) computer program. It briefly describes the mathematical problem solved by FEARS. It tells how to prepare the input data for FEARS and how to interpret the output provided by FEARS. A composite shaft problem is provided as an example.

ADMINISTRATIVE INFORMATION

This work was performed under David W. Taylor Naval Ship R&D Center's (DTNSRDC) Independent Research Program, Program Element 61152, Task Area ZR0140201, DTNSRDC Work Unit 1844-140. Professor Ivo Babuska of the Institute for Physical Science and Technology (Contract N00167-82M-0743) and Mr. Charles K. Mesztenyi of the Computer Science Department, University of Maryland (Contract N00167-82M-0636) are contractors to DTNSRDC.

1. INTRODUCTION

The objective of the Finite Element Adaptive Research Solver (FEARS) project was to design an innovative finite element solver and assess its feasibility and computational efficiency. Once completely developed, this new finite element solver would have the following characteristics:

- It would produce reliable information about the total accuracy of the finite element model. This information would take into account both the error due to the finite element discretization and the error of the model itself (i.e., the errors caused by the small strain assumption, the linear constitutive law, etc.).

- It would require the user to supply only the necessary physical information, the computational aims, and possible restrictions. For example, the user might want to compute the maximum stresses or the stress intensity factor, say with an accuracy to within about 5% and with certain restrictions pertaining to computer cost, memory availability, etc.

- It would be completely automated, user-friendly, and flexible, allowing the user to perform such studies as sensitivity analyses and evaluations of engineering data.

- It would reflect the recent state of the art in finite element technology as well as current trends in mathematics, computer science, and engineering.

The present version of FEARS implements some of these basic features such as an error estimate for the finite element discretization and an adaptive approach to mesh refinement. It also provides an option for postprocessing. This option will enable the user to evaluate both the mathematical model used and the reliability of the engineering data and to treat problems in fracture mechanics among other things when the appropriate postprocessors are implemented.

The design of the FEARS finite element solver is unique, and it has capabilities which make it significantly different from all present finite element codes. Section 2 of this report briefly discusses the main features of FEARS. Section 3 describes the problem of a cracked composite shaft, formulates the engineering problem, and states the problem in the form directly solvable by FEARS (the F-problem). Section 4 describes the solution of the F-problem by FEARS, including preparation of the data for the FEARS input of the F-problem, the input in detail, and the FEARS output for the F-problem. Section 5 summarizes some results relevant to the adaptive aspect of FEARS.

The basic mathematical theory underlying the FEARS program was developed at the University of Maryland. The FEARS computer program was also designed and implemented there. A numerical investigation of certain problems of special interest to the Navy is underway at DTNSRDC, using the FEARS program.

2. MAIN FEATURES OF FEARS

The current FEARS program has the following features:

- It has eight mesh refinement procedures which are used in an adaptive (or interactive) sequence to provide a solution with an accuracy specified by the user. This solution is the most accurate one under the constraints (cost, time, etc.) imposed by the user.
- It provides reliable estimates of the error of the finite element solution with respect to the (unknown) exact solution of the given boundary value problem.
- Its input is both simple and basic. It requires only the necessary physical and geometric data.
- It has an option for restarting the computation when interrupted by the user or by one of the various stopping criteria.
- It provides a flexible frame for the various postprocessing procedures.

FEARS solves boundary value problems for a linear self-adjoint elliptic system of partial differential equations of second order with two field variables in two dimensions when these boundary value problems are formulated in a weak form on a domain $\Omega \subset R^2$. The following discussion illustrates the features of FEARS in more detail.

a. DOMAIN Ω

The domain Ω is the union of the interiors of a small number (usually $\leq$ 16) of curvilinear (open) rectangles bounded by circular (or straight) arcs. These rectangles are called 2-D domains and are denoted by D_i^2, $i = 1,\ldots,N_2$. The (open) bounding arcs are called 1-D domains and are denoted by D_j^1, $j = 1,\ldots,N_1$. The vertices of the 2-D domains are called 0-D domains and are denoted by D_k^o, $k = 1,\ldots,N_o$.

b. PROBLEMS SOLVED (WEAK FORM)

FEARS solves problems formulated in the following weak form:

$$\sum_i \int_{D_i^2} \left\{ \left[\frac{\partial V}{\partial Z}\right]^T A_i \left[\frac{\partial U}{\partial Z}\right] + V^T B_i \left[\frac{\partial U}{\partial Z}\right] + \left[\frac{\partial V}{\partial Z}\right]^T B_i^T U + V^T C_i U \right\} dx\, dy$$

$$+ \sum_j \int_{D_j^1} V^T \gamma_j U\, ds =$$

$$\sum_i \int_{D_i^2} \left\{ \left[\frac{\partial V}{\partial Z}\right]^T D(x,y) + V^T E(x,y) \right\} dx\, dy \qquad (2.1)$$

$$+ \sum_j \int_{D_j^2} \varepsilon_j(s)\, V\, ds$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \frac{\partial U}{\partial Z} = \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial y} \end{bmatrix}$$

and analogously for $\frac{\partial V}{\partial Z}$ and V.

Further

A_i is a 4 x 4 symmetric (constant) matrix

B_i is a 2 x 4 (constant) matrix

C_i is a 2 x 2 symmetric (constant) matrix

γ_j is a 2 x 2 symmetric (constant) matrix

$D_i(x,y)$ is a 4 x 1 vector valued function

$E_i(x,y)$ is a 2 x 1 vector valued function

$\varepsilon_j(s)$ is a 1 x 2 vector valued function (s is arc length).

For a discussion of the physical significance of the matrices, the reader is referred to the User's Manual.[1*] The trial and test spaces used allow us to handle essential and natural boundary conditions. The present formulation allows us to deal with materials which are homogeneous on every 2-D domain and with boundary conditions that are of the same type on every 1-D domain.

c. ERROR NORMS

The admissible error norm for the a-posteriori estimation of the error $e = u_{FE}-u_o$ (where u_{FE} and u_o denote the finite element and exact solution, respectively) is $|||e|||_{2p}$.

---

* The reference is given on page 33.

$$|||e|||_{2p} = \left[\sum_i \int_{D_i^2} \left[\frac{\partial e}{\partial Z}^T (A_E)_i \frac{\partial e}{\partial Z} \right]^p dxdy \right]^{\frac{1}{2p}} \tag{2.2}$$

If $(A_E)_i = A_i$, $p = 1$, $B_i = 0$, $C_i = 0$, and $\gamma_j = 0$, then $|||.|||_{2p}$ is the strain energy (but multiplied by 2). Because of imposed boundary conditions, $|||.|||_{2p}$ is not a seminorm but a norm.

d. BOUNDARY CONDITIONS AND RIGHT-HAND SIDES

FEARS provides a variety of admissible boundary conditions and right hand sides. They can be defined either in terms of a class of functions characterized by a set of parameters or through user-provided subroutines.

e. ELEMENTS

FEARS uses elements of a bilinear type on curvilinear rectangles. (The elements are bilinear when mapped onto the master square). Under certain general conditions the asymptotic accuracy for a smooth solution is of order h with a rate of convergence of $N^{-1/2}$ for a quasi-uniform mesh with respect to the energy norm (p = 1). With respect to the L_2 norm the asymptotic accuracy is of order h^2 with a rate of convergence of N^{-1} (h denotes the size of the elements used and N the number of degrees of freedom). The postprocessing allows us to compute the functionals (e.g., the stresses at points of interest) with an accuracy of order h^2 and a rate of convergence of N^{-1} (energy norm). The adaptive construction of the meshes leads to the same rate of convergence with respect to N whether the solution is smooth or has singular behavior. (The quasi-uniform mesh can result in a rate of covergence $N^{-\varepsilon}$ with $\varepsilon > o$ arbitrarily small if the singularity of the solution is very strong.)

f. STORAGE STRUCTURE

The storage structure of the FEARS program is different from that of the usual finite element program due to its adaptive features. It is based on a quartic rooted tree structure (father -> 4 sons). Gaussian elimination is used

to solve the systems of linear equations providing the finite element solution. The process is parallel and independent with respect to 2-D domains. This feature is reflected in the storage segmentation structure. The micro-stiffness computations, the collection process, and the LU decomposition are performed simultaneously on the 2-D domain level. The data structure for the adaptive construction of the meshes avoids the necessity for reordering the unknowns (for the purpose of elimination) because it is an optimal or nearly optimal one. The unknowns located on the 1-D and 0-D domains are eliminated by a sparse (minimal degree) algorithm.

g. ERROR ESTIMATORS AND INDICATORS

The computation of the error estimator is based on the computation of the error indicators of each element, which are computed independently on every 2-D domain. Optimal meshes are characterized by the equilibration (equidistribution) of the indicators.

h. MESH REFINEMENT

FEARS provides the user with eight modes of mesh refinement. This allows the user to create arbitrary meshes or have meshes adaptively constructed. The complete solution of the finite element equations for the given mesh, including computation of the error indicators and estimators, is called the "long path" solution. The "short path" solution provides the solution on various subregions using the nodal solution values outside of these regions obtained by previous computation on coarser mesh. The "short path" solution on the element level is inexpesive, and the excessive use of "long path" solutions is to be avoided.

i. DUMP FILE

FEARS provides various dump file possibilities which allow the resumption of computation if processing is terminated by the user or by one of the stopping criteria.

j. OUTPUT OPTIONS

FEARS affords a variety of output options with respect to the amount of information to be printed. It also provides information about time used and

the major characteristics of the adaptive steps such as statistical distributions of element sizes, sizes of the error indicators, etc.

k. s VECTOR

FEARS allows the user to compute and print out the five dimensional vector $s = (OUTP(1), OUTP(2), \ldots, OUTP(5))^T$

where
$$s = S \begin{bmatrix} \frac{\partial U}{\partial Z} \\ U \end{bmatrix} \tag{2.3}$$

and S is a 5 x 6 matrix. The values of the vector s are computed at the centers of the elements. For elasticity problems $s = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, u_1, u_2)^T$ where σ_{xx}, σ_{yy}, σ_{xy} are the stresses and u_1, u_2 the displacements.

l. POSTPROCESSING

FEARS provides a frame for postprocessing which will allow the evaluation of various functionals providing stresses, stress intensity factors, etc. with higher accuracy than usual as well as error estimates.

3. PROBLEM OF THE CRACKED COMPOSITE SHAFT

a. PROBLEM DESCRIPTION

The cracked composite shaft is shown in Figure 3.1. The inner disk is steel (Young's modulus of elasticity, $E = 3.\text{x}10^7$ psi, Poisson's ratio, $\nu = .3$) and the outer ring is bronze ($E = 1.6\text{x}10^7$ psi, $\nu = .32$). The crack is at a 45-degree angle to the horizontal x-axis and extends halfway through the bronze ring (i.e., $\rho = r_o + .5R$). Plane strain behavior is assumed. The composite shaft is stress free when the temperature of the inner steel disk $T_S = 0$ and that of the outer bronze ring $T_B > 0$. Moreover, on the interface of the two no slippage can occur. The problem is then to determine the stress state when, after cooling, $T_B = T_S = 0$. Obviously tension will occur in the bronze ring and the crack will open up inside, although both ends of the crack stay closed (due to non-slippage). An independent postprocessing capability for FEARS will

be developed which can compute the stress intensity factors at both ends of the crack.

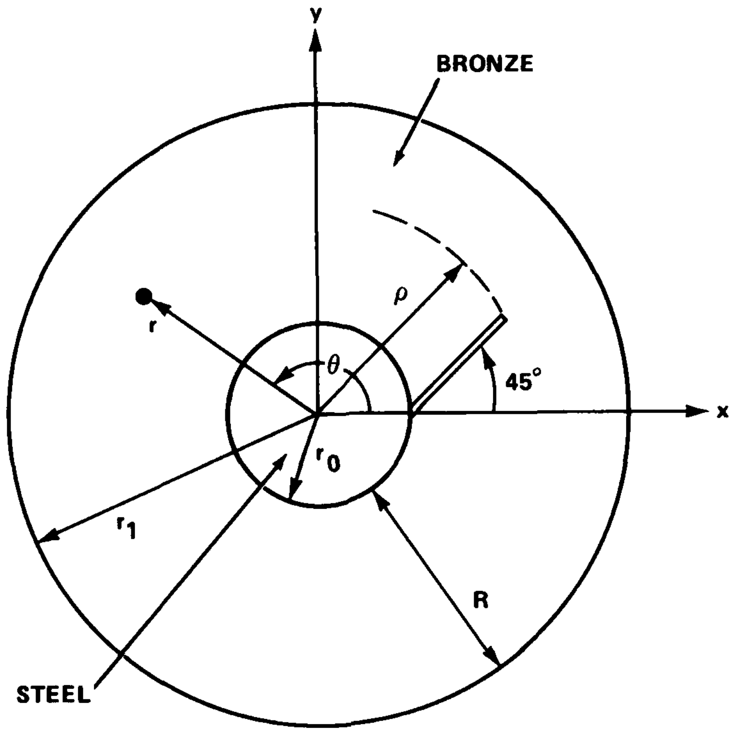

Figure 3.1 - Cracked Composite Shaft

b. F-PROBLEM

The problem must be cast in a form which FEARS can solve, and this FEARS-solvable problem is called an F-problem.

The F-problem: Find the stress state (plane strain) of the cracked composite shaft when the internal interface of the steel disk and bronze ring is loaded with a unit radial constant load. The accuracy is to be within about 5% measured in the energy norm. (The energy norm of the error is the square root of the absolute value of the difference between the energies of the exact and finite element solutions of the problem.) The F-problem is essentially equivalent to the cracked composite shaft problem in the sense that, when the solution of the F-problem is known, the solution of the original problem is easily obtained. If the outer bronze ring is allowed to cool (unimpeded by the inner steel disk) to the temperature T=0, the inner boundary is displaced

by an amount $\Delta = T_B r_o$, where α is the thermal expansion coefficient for bronze. On the other hand a negative radial load $Z = \frac{\alpha T_B G_s}{(1-2\nu_s)}$ placed on the interface creates a hydrostatic pressure in the steel disk and produces a displacement Δ in the outer boundary of the disk. Therefore, this load in the original problem results in no stress in the bronze ring and a hydrostatic stress state in the steel disk. The additional stress caused by the removal of the auxiliary load Z is then computed by the F-problem. This removal is effected by applying the positive radial load Z to the steel-bronze interface. The combination of the hydrostatic stress in the steel disk and the stress state computed by FEARS for the F-problem yields the solution to the original shrink fit problem. (See Figure 3.2)

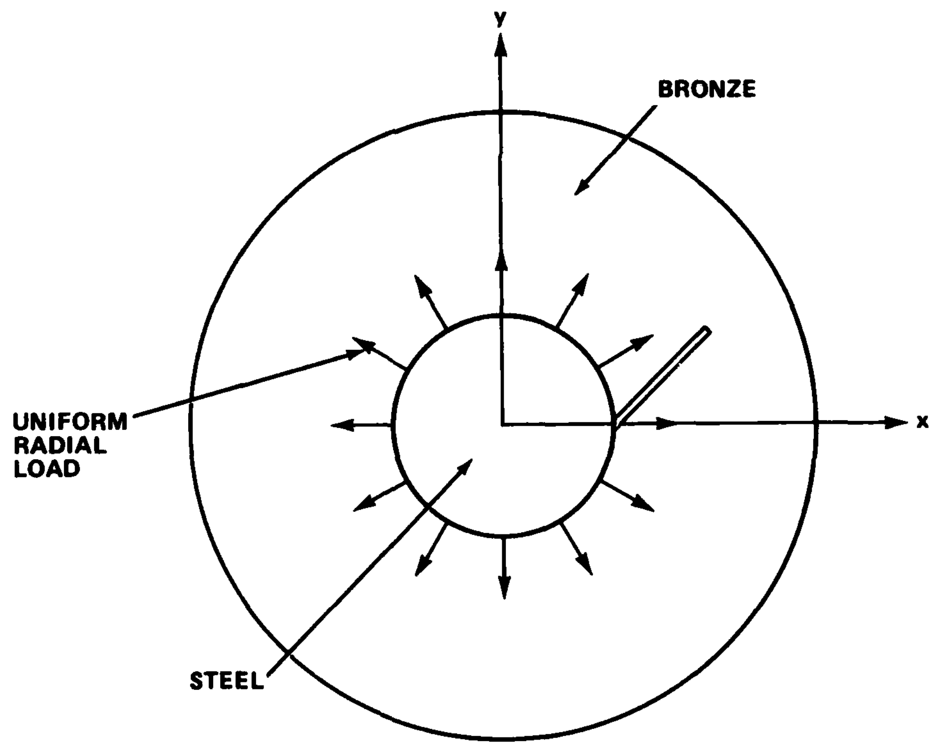

Figure 3.2 - Scheme of the F-Problem

4. FEARS SOLUTION OF THE F-PROBLEM

The FEARS solution consists of the following parts:

- Data preparation
- Input
- Output

These stages are common to all FEARS solutions.

a. PREPARATION OF THE DATA

The essential data of the F-problem consist of

- Geometry
- Material properties
- Boundary conditions and loads
- Output computation.

(1) Geometry

The domain must be partitioned into a set of curvilinear quadrilaterals bounded by circular arcs or straight line segments. Some restrictions are placed on the shape of the quadrilaterals with respect to the angles at the vertices and the angles of the circular arcs. (See the User's Manual.[1]) It is essential for the F-problem of our example that

- The bronze-steel interface coincides with 1-D domains.
- The sides of the crack are distinct 1-D domains and the ends of the crack are 0-D domains.
- Every 2-D domain consists entirely of bronze or of steel.

Figure 4.1 shows the selected partition.

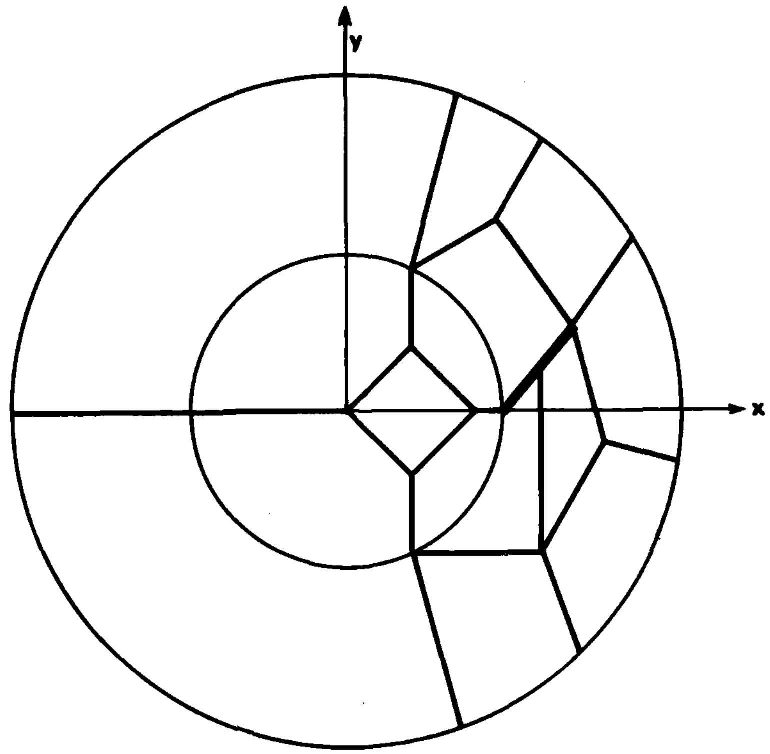

Figure 4.1 - The Partition of the Domain

(2) Material Properties

The bilinear form of Equation (2.1) expresses the usual principle of virtual work. Accordingly, in this case $B = C = \gamma = 0$ and

$$A = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 0 & 0 & \nu \\ 0 & \frac{1-2\nu}{2} & \frac{1-2\nu}{2} & 0 \\ 0 & \frac{1-2\nu}{2} & \frac{1-2\nu}{2} & 0 \\ \nu & 0 & 0 & 1-\nu \end{bmatrix} \tag{4.1}$$

The matrices A for steel and bronze are different. For steel $E_S = 3. \times 10^7$ psi, $\nu_S = .3$, and for bronze $E_B = 1.6 \times 10^7$ psi, $\nu_B = .32$.

(3) Boundary Conditions and Loads

The loads are defined through the right hand side of Equation (2.1). Because the load is on the bronze-steel interface only, D = E = 0 and the right hand side consists only of the term

$$\sum_j \int_{D_j^1} (v_1 \cos\phi \; v_2 \sin\phi)\, ds$$

where the integration is taken over all 1-D domains on the bronze-steel interface. ϕ is the polar angle. (FEARS allows direct input of radial loads.)

There is no restriction on the u_i except for the specification of displacement for two 0-D domains to prevent rigid body motion.

(4) Output Computation

The desired stresses and displacements are given by Equation (2.3) where

$$S = \left[\begin{array}{cc} \frac{1}{(1+\nu)(1+2\nu)} \begin{bmatrix} 1-\nu & 0 & 0 & \nu \\ \nu & 0 & 0 & 1-\nu \\ 0 & \frac{1-2\nu}{2} & \frac{1-2\nu}{2} & 0 \end{bmatrix} & \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \\ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} & \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \end{array} \right] \qquad (4.2)$$

and $s = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, u_1, u_2)^T$

b. INPUT

The input for the FEARS program is fairly simple and is divided into the following seven groups:

- Job control language
- Initial input
- Problem header
- Geometry input
- Various matrices
- Initial subdivision input
- Command input

Table 4.1 (pages 14-16) lists the input for the cracked composite shaft problem discussed in Sections 2 and 3 and displays these seven input groups which are described in detail in the paragraphs that follow. This problem and its input are fairly standard, but other options are available for different types of problems. The FEARS User's Manual[1] should be consulted for complete descriptions of all these options.

(1) Job Control Language

The job control language of Table 4.1 is that of the Cyber 7400 at DTNSRDC under the NOS/BE operating system. The present version of FEARS requires a field length of CM230000, which must be specified on the job card. This field length does not allow P4 priority (prime express) runs, but one can use P3 priority (prime batch) if the run will take less than 600 CPU seconds.

The present version of FEARS resides in the permanent file ABSFXYO, ID=PUMZ. The request for the permanent file DUMP as well as its subsequent cataloging under the user's ID is needed because of the DUMP command (see paragraph (7)). FEARS writes out the computed results on this file which subsequently can be used as input either for restarting FEARS or for future postprocessors.

(2) Initial Input

The initial input consists of two lines. The first line, consisting of eight integers, specifies that the input data and the output produced by each interactive step will be printed on tape 6. See the User's Manual[1] for the various options available for the disposition of output. The second line consists of one integer. The value zero indicates a start from the beginning. For the details of the restart procedure the reader is referred to the User's Manual.[1]

(3) Problem Headers

The problem header consists of three lines. The first line, consisting of the two entries 1,0., indicates that there is no optional user-supplied subroutine for this problem. The second line, consisting of eight digits, is the identification number supplied by the user. The third line consists of one real number p = 1.0 which indicates the user's choice of norm.

TABLE 4.1 - INPUT LISTING FOR THE CRACKED COMPOSITE SHAFT PROBLEM

JOB CONTROL LANGUAGE			JOB CARD
			CHARGE CARD
			ATTACH.ABS.ABSFXYO,ID=PUMZ.
			REQUEST,DUMP,*PF.
			ABS.
			CATALOG,DUMP,COMPOSEDSHAFT,ID=USER.
			EOR
INITIAL INPUT			0.0.0.0,0,0.1.0
			0
PROBLEM HEADER			1.0.
			11000000
			1.0
GEOMETRY	0–D DOMAINS		20
		n_1	1,-4.,0..3.0..0.
		n_2	2,-2.,0.,0.0..0.
		n_3	3.0..0..0.0..0.
		n_4	4,1.6.3.666060556.2.0..0.
		n_5	5,.8,1.833030278.0,0..0.
		n_6	6..8..8,0.0..0.
		n_7	7.2.6.3.039736831,0,0..0.
		n_8	8,2.,2.2,0.0.,0.
		n_9	9,3.645751311,1.645751311.0.0..0.
		n_{10}	10,2.870828693,.8708286935,0.0..0.
		n_{11}	11,2.435414347..4354143465.0.0.,0.
		n_{12}	12.2..0..0.0.,0.
		n_{13}	13,1.4.0.,0,0..0.
		n_{14}	14.3.919183588,-.8,0,0..0.
		n_{15}	15,3.4,-.8.0,0.,0.
		n_{16}	16.2.435414347.-1.8,0.0..0.
		n_{17}	17,2.8.-2.856571371,0.0..0.
		n_{18}	18.1.6.-3.666060556,0.0..0.
		n_{19}	19..8.-1.833030278,0.0..0.
		n_{20}	20,.8,-.8,0,0.,0.

TABLE 4.1 - (Continued)

GEOMETRY	1–D GEOMETRY		35
		s_1	1.1,4,0,.25
		s_2	2,4,7,0,.25
		s_3	3,7,9,0,.25
		s_4	4,9,14.0..25
		s_5	5.14.17,0,.25
		s_6	6.17,18.0,.25
		s_7	7.18.1.0,.25
		s_8	8,2,5.0..5
		s_9	9.5,12,0,.5
		s_{10}	10.12.19,0,.5
		s_{11}	11,19.2.0..5
		s_{12}	12,3,6,0,0.
		s_{13}	13,6.13,0,0.
		s_{14}	14.13,20.0.0.
		s_{15}	15,20.3.0.0.
		s_{16}	16,1.2,0,0.
		s_{17}	17,2,3.0.0.
		s_{18}	18,4.5,0,0.
		s_{19}	19,7.8.0.0.
		s_{20}	20.8,5,0.0.
		s_{21}	21.5,6.0.0.
		s_{22}	22.8.10.0.0.
		s_{23}	23,9,10.0.0.
		s_{24}	24.10.12.0,0.
		s_{25}	25.10.11,0,0.
		s_{26}	26,11,12.0,0.
		s_{27}	27.12.13.0,0.
		s_{28}	28,14,15.0.0.
		s_{29}	29,10,15,0,0.
		s_{30}	30,15.16.0,0.
		s_{31}	31.11,16.0.0.
		s_{32}	32,16.17,0,0.
		s_{33}	33.16.19.0,0.
		s_{34}	34.18.19.0.0.
		s_{35}	35.19,20,0,0.
	2–D GEOMETRY		15
		d_1	1.2.1.5.4
		d_2	2.3.2.6.5
		d_3	3,19.18.2.1
		d_4	4.20.19,3,2
		d_5	5.20.3.13.6
		d_6	6.5.4,8.7
		d_7	7,8.7,10,9
		d_8	8,5.8.12,10
		d_9	9.6.5.13.12
		d_{10}	10.10.9,15.14
		d_{11}	11.11.10.16.15
		d_{12}	12,12.11.19,16
		d_{13}	13,13.12.20.19
		d_{14}	14.15.14.16,17
		d_{15}	15,16.17.19,18

TABLE 4.1 - (Continued)

Section	Subsection	Package	Item	Input
THE MATRICES	MATRICES OF 2-D DOMAINS		ℓ_1	2
		PACKAGE 1 [BRONZE]	ℓ_2	1,10.1,3.6.7.8.10.11.12.14,15
			ℓ_3	1.0.0.0.0
			A	22895622.9,0..0.,6060606.061,
				0.,10774410.77,6060606.061,0..
				0..6060606.061.10774410.77,0..
				6060606.061,0..0.,22895622.9
			A_E	22895622.9,0..0.,6060606.061,
				0.,10774410.77.6060606.061.0..
				0,6060606.061,10774410.77,0..
				6060606.061.0..0.,22895622.9
			ℓ_4	1..1..1..1.
			S	22895622.9.0..0..10774410.77,0.,0..
				10774410.77,0..0..22895622.9.0..0..
				0..6060606.061.6060606.061.0..0..0..
				0..0..0..0..1..0..
				0..0..0..0..0..1.
		PACKAGE 2 [STEEL]		2,5,2,4,5.9,13
			ℓ_3	1,0,0,0.0
			A	40384615.38,0..0..11538461.54,
				0.,18939393.94,11538461.54.0..
				0.,11538461.54,18939393.94.0..
				11538461.54.0..0.,40384615.38
			A_E	40384615.38.0..0..11538461.54,
				0.,18939393.94,11538461.54,0..
				0.,11538461.54.18939393.94.0.,
				11538461.54.0..0..40384615.38
			ℓ_4	1..1..1..1.
			S	40384615.38.0..0..18939393.94.0..0..
				18939393.94.0..0..40384615.38.0..0..
				0..11538461.54,11538461.54,0..0..0..
				0..0..0..0..1..0..
				0..0..0..0..0..1.
	MATRICES OF 1-D DOMAINS		ℓ_1	1
			ℓ_2	1.4.8,9.10.11
			ℓ_3	0,-1
			ℓ_4	0..0..0..0.
			ℓ_5	0..-1..0..-1.
INITIAL SUBDIVISION				1
COMMAND INPUT				ITERATE
				6..2,600...05
				DUMP
				20
				PRINT
				-1.-1,3
				OUTPUT
				1
				STOP

(4) Geometry Input

This group deals with the geometry of the problem. The domain is divided into two-dimensional (2-D), one-dimensional (1-D), and zero-dimensional (0-D) subdomains as shown in Figure 4.2. The 0-D, 1-D, and 2-D domains are numbered independently, starting with 1.

(a) <u>0-D Domain (Node) Data</u>. The first line contains the number m of the 0-D domains. Then the lines n_i, $i = 1,\ldots,m$ describe the i^{th} 0-D domain. Each line n_i consists of six entries:

$$i, x, y, b, u, v$$

i is the index number of this point, x and y are its coordinates, b is its boundary condition number (integer), and u and v are the prescribed values of the enforced displacement vector of this point. The values which b can assume, as well as the conventions for u and v, are described in the User's Manual.[1] In the present example only points 1 and 4 have prescribed displacements.

(b) <u>1-D Domain (Side) Data</u>. The first line contains the number q of the 1-D domains. Then the lines s_i, $i = 1,\ldots,q$ describe the i^{th} 1-D domain (side). Each line s_i consists of five entries:

$$i, p, q, c, k$$

i is the index number of the line, p and q are the indices of the end points of this 1-D domain, c is its boundary condition number (integer), and k is the curvature of this 1-D domain. Conventions regarding the orientation of the 1-D domain, the sign of its curvature, and the values of c are described in the User's Manual.[1]

(c) <u>2-D Domain Data</u>. The first line contains the number j of the 2-D domains. Then the next j lines d_i, $i = 1,\ldots,j$ describe the i^{th} 2-D domain. Each line d_i consists of five entries:

$$i, p, q, r, s$$

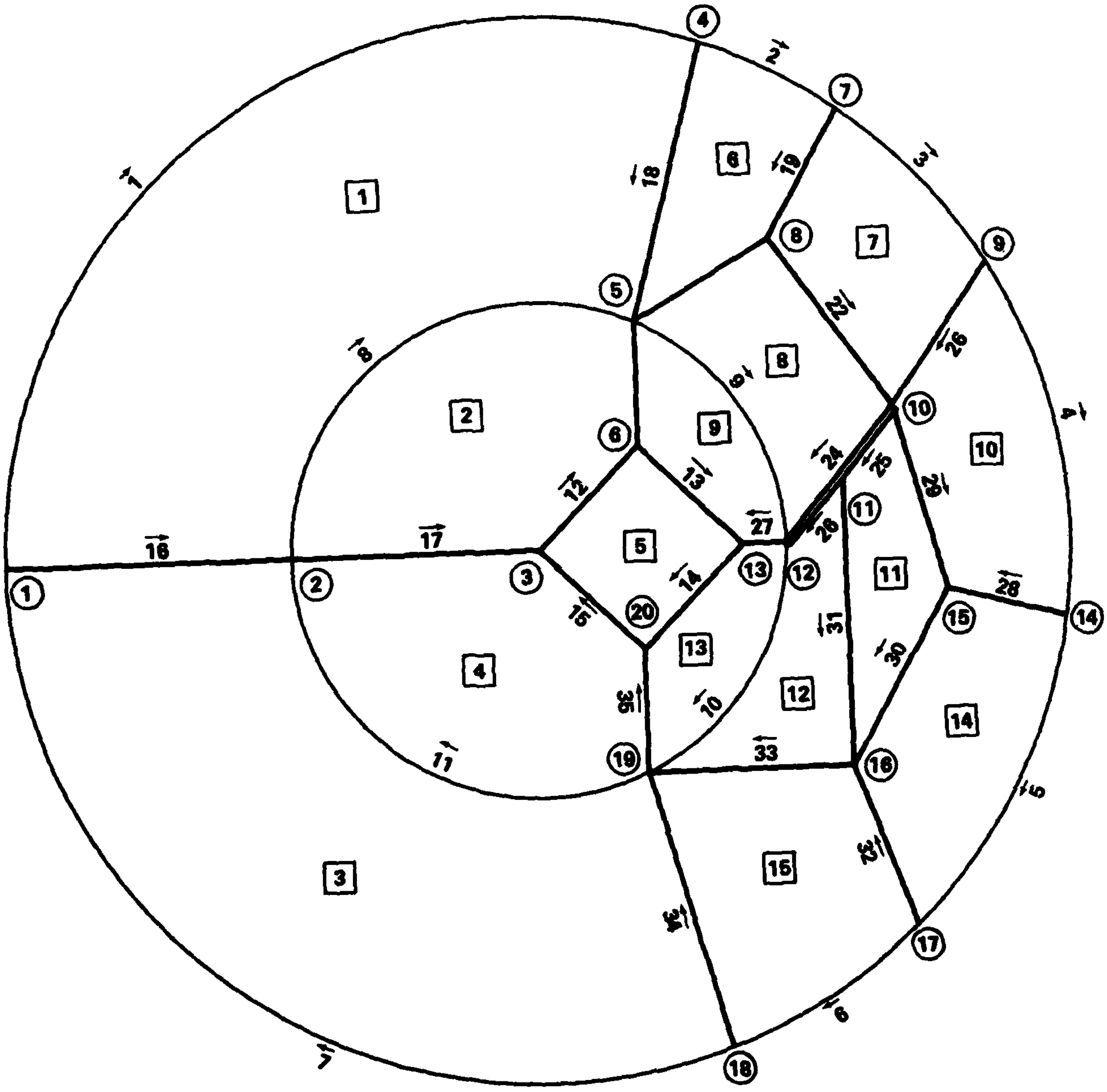

Figure 4.2 - Numbering of the Domains

i is the index number of the 2-D domain and p, q, r, and s are the indices of its four corner points. The ordering of the indices of the corner points must be consistent with respect to clockwise orientation, since the ordering for the "master square" is

$$\begin{matrix} q' & s' \\ p' & r' \end{matrix}$$

An admissible ordering is such that by a rotation it can be brought into agreement with the ordering of the "master square". For example,

$$\begin{matrix} r' & p' \\ s' & q' \end{matrix}$$

is admissible, but neither of the following two orderings is.

$$\begin{matrix} q' & p' \\ r' & s' \end{matrix} \quad \text{or} \quad \begin{matrix} q' & r' \\ s' & p' \end{matrix}$$

See the User's Manual[1] for more details. Alternatively the FEARS program allows the 2-D domains to be input in terms of their sides (which are 1-D domains) instead of their corner points. (The present example specifies its 2-D's by their corner points.)

(5) Various Matrices

This group contains the matrices of the bilinear form (Equation 2.1), the matrices of the norm (Equation 2.2), and the output matrix (Equation 2.3).

(a) Matrices of the 2-D Domains. FEARS permits the 2-D domains to have different sets of matrices which are input in packages as follows:

The first line of the group consists of the number of different packages of matrices ($\ell_1 = 2$). Then the different packages are listed one after another. The first line (ℓ_2) of a package has the form

$$i,\ n,\ d_1,\ d_2,\ldots,d_n$$

where i is the index of the package, n is the number of the 2-D domains to which this package of matrices applies, and the d_i's are the index numbers of those 2-D's. Line ℓ_3 consists of five integer entries which indicates whether

the matrices A, B, C, D, E are present or not. (In this example only the matrix A is present.)

The next four lines in the package contain the matrix A as given by Equation (4.1). This matrix A must be entered according to the numbering scheme

1	3	9	11
2	4	10	12
5	7	13	15
6	8	14	16

The next four lines describe the matrix A_E for the error norm. The reader should also note that in this case $A_E = A$. A_E is entered in the same fashion as A. Data are read by the FEARS program in "free format", i.e., without reference to fixed FORTRAN FORMAT statements. Accordingly, A and the other matrices could have been input using any number of consecutive lines of data.

The next line ℓ_4 in the package consists of four numbers pertaining to the computation of the norm. See the User's Manual[1] for the details. At the end of the package is the output matrix used to compute the stresses and displacement. The output matrix S, given by Equation (4.2), is a 5 x 6 matrix consisting of three stress components and two displacements. It is input according to the following scheme:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

Again the reader should note the flexibility afforded by "free format" as regards matrix input.

This completes the input for the first package which describes the bronze outer ring of the present example. The input for the remaining 2-D domains pertaining to the inner steel disk is prepared in a similar way.

(b) <u>Matrices of the 1-D Domains</u>. This group contains the input for the line integration, also broken up into packages. The first line of the group consists of the number of packages ($\ell_1 = 1$). Then the different packages are listed one after another. The first line (ℓ_2) of a package is of the form

$$i, n, \rho_1, \rho_2, \ldots, \rho_n$$

where i is the index of the package, n is the number of 1-D domains to which the package applies, and the ρ_i are the indices of those 1-D domains. The line ℓ_3 consists of two numbers

$$g, \ell$$

where g indicates whether line integration occurs with respect to the bilinear form, and ℓ (integer) indicates what type of line integration is required by the right hand side of Equation (2.1). In this case the polar load is indicated by $\ell = -1$. The line ℓ_4 consisting of four entries relates the line integration to the bilinear form. If line ℓ_3 indicates that no line integration is to take place, then line ℓ_4 must contain appropriate dummy values. The line ℓ_5 consists of four entries describing the load vector with respect to polar coordinates at the end points of the 1-D domain. For more details, see the User's Manual.[1]

(6) Initial Subdivision Input

This group, consisting of one line of one integer entry, indicates whether the initial finite element mesh has four or sixteen elements per 2-D domain. The present example begins with the quartering of the 2-D domains.

(7) Command Input

The commands, which are English words and may be abbreviated by the first four letters of the word, are executed by the FEARS program in the order given in the data deck. Most of these commands must be followed by one or two lines of "free format" entries which are the arguments of that command. Only four commands (ITERATE (ITER), PRINT (PRIN), DUMP, and STOP) are considered here. The words in parentheses are the acceptable shortened form of the command.

The ITERATE command directs the FEARS program to solve the problem of the data deck using its automated iterative procedure. The ITERATE command is followed by a line of four arguments

$$m, \beta, t, \alpha$$

which specify the conditions under which the program will terminate. (Of course, the computation will be aborted if more memory is called for than is available.) As soon as one of the termination conditions is satisfied, the

iterative procedure is halted and the FEARS program goes to the next command. Here m is the maximum number of interactive steps to be performed, β is a constant determining the combination of long and short paths, t (real number) is the maximum allowable number of CPU seconds for the computation, and α is the required relative accuracy (if applied to positive definite problems with respect to energy norms as in the case of elasticity problems). 0.3 has been found by experience to be a good value for β. If the value of β is zero, then all the iterative steps are long paths.

The DUMP command is followed by an integer argument indicating the number of the tape unit on which this FEARS dump is to be written, in this case unit 20. This DUMP command causes a permanent file to be written and enables the user to restart the computation from the iteration step just before the DUMP command.

The PRINT command is followed by a line of three integer arguments

a, b, c

a determines which type or types of domains are to be printed out, b indicates what sort of information is desired, and c specifies how much information is to be printed. The most commonly used values for these parameters are a = -1, b = -1, and c = 3; these provide all essential information.

The OUTPUT command directs FEARS to calculate and print out the stresses. It is followed by a line containing the integer 1. Finally, the STOP command terminates the execution of FEARS.

c. OUTPUT

The FEARS program provides the user with many options for the output. Only those forms of the output which the average user would encounter are described here. They are:

- Main summary output. This is the information provided by one iterative step (either a "long path" or "short path" solution) of the automated solution procedure.
- Total output. This is the information printed by the

```
PRINT
-1,-1,3
```

command. It contains all the essential information concerning the mesh and the solution at a given step of the automated solution procedure.

- s output. s is defined by Equation (2.3). The components of the vector s are denoted by OUTP(1),...,OUTP(5).

(1) Main Summary Output

The main output summary for each iteration step begins with a display similar to the one given here.

```
=== ESTIMATES FOR ITERATION STEP  1
 2-D          ELEMENTS        POINTS     STORAGE
INDEX     PRES.    SUBD.      PRES.     DATA        STACK
  1         4        2          1             12          88
  2         4        0          1              5          40
  3         4        2          1             12          88
  4         4        0          1              5          40
  5         4        0          1              5          40
  6         4        0          1              5          40
  7         4        0          1              5          40
  8         4        0          1              5          40
  9         4        0          1              5          40
 10         4        0          1              5          40
 11         4        0          1              5          40
 12         4        0          1              5          40
 13         4        0          1              5          40
 14         4        0          1              5          40
 15         4        0          1              5          40
RESULTING IN   72 ELEMENTS WITH MATRIX STORAGE      3586   AND LONG PATH TIME    .182808E+0
```

This display contains information concerning the present mesh and number of suggested elements for refinement in the next iteration step along with some storage information. Ordinarily the user will not be concerned with this display since it may not describe the actual refinement that is effected at this iteration step. (This display may not be printed in future versions of the FEARS program.)

The next display presents information about the actual mesh refinement effected at this iteration step as well as information concerning the refinement of this present mesh at the next iteration step. Information is also provided about the solution calculated for the present mesh. Here is a sample of such a display.

```
                     LONG PATH EXECUTION

===== F U L L   D O M A I N =====
NUMBER OF POINTS-    76      NUMBER OF ELEMENTS-    72
                    BDRY POINTS-    55
ENERGY NORM-   .601353E-03      ENERGY-    .361624978738E-06
ERROR ESTIMATOR-   .866949074996E-04      RELATIVE ERROR-   .144167E+00
MAX.ERROR INDICATOR-   .901867667558E-09      BY 2-D INDEX-  4      THRESHOLD-   .459925E-09
APPROXIMATE NUMBER OF ELEMENTS TO BE SUBDIVIDED-
 2-D   TO BE SUBD.    PRESENT     (POINTS)
  1        0            10           4
  2        2             4           1
  3        0            10           4
  4        2             4           1
  5        0             4           1
  6        0             4           1
  7        0             4           1
  8        0             4           1
  9        0             4           1
 10        0             4           1
 11        0             4           1
 12        0             4           1
 13        0             4           1
 14        0             4           1
 15        0             4           1
 TOTAL     4
STORAGE SIZES=
   2-D MATRIX, MAX.CORE =   112,   TOTAL =       800,  NO RECORDS =   19
   BDRY MATRIX =      3143
EXECUTION TIME=
   SUBDIVISION=              .3730E+00
   2-D MATRIX SOLUTION=      .4075E+01
   BDRY MATRIX SOLUTION=     .2359E+01
   2-D ERROR CALCULATION=    .1585E+01
BDRY MATRIX UPDATES      1070 IN   1 RECORDS
```

The number of points is the number of nodes in the 0-D, 1-D, and 2-D domains which have at least one degree of freedom. The number of elements is the total number of elements in the full domain. The boundary points ("BDRY POINTS") entry is the number of points on the 0-D and 1-D domains which are not completely fixed. The energy norm is the square root of the value of the energy when the bilinear form expressing the energy is used. The error estimator gives the error estimate for the full domain. The relative error is the energy norm divided by the error estimator. The next line gives the maximum error indicator, the index of the 2-D domain in which it occurs, and the threshold value for automatic refinement. (All elements with error indicators above the threshold were refined.) Then comes a list of the number of elements in each 2-D domain which will be subdivided if the threshold value is the criterion for subdivision. Finally some storage information and a breakdown of the computing time for that step is provided.

(2) Total Output

The total output is produced by the

PRINT
-1,-1,3

command. This output contains detailed information about the 0-D, 1-D, and 2-D domains. Part of one such display is reproduced here.

```
--- 0-D        COORDINATES                   SOLUTION                      ERROR                 BDRY  EXT
 1  -.400000E+01 0.             0.            0.            0.            0.                3    1
 2  -.200000E+01 0.             .105294E-07   .677532E-08   .105294E-07   .677532E-08       0    0
 3  0.           0.            -.193643E-07   .130282E-07  -.193643E-07   .130282E-07       0    0
 4   .160000E+01  .366606E+01  -.369700E-07  0.            -.369700E-07  0.                 2    1
 5   .800000E+00  .183303E+01  -.360329E-07  -.121135E-07  -.360329E-07  -.121135E-07       0    0
 6   .800000E+00  .800000E+00  -.332789E-07   .363740E-08  -.332789E-07   .363740E-08       0    0
 7   .260000E+01  .303974E+01  -.402988E-07   .599245E-08  -.402988E-07   .599245E-08       0    1
 8   .200000E+01  .220000E+01  -.401287E-07   .155474E-08  -.401287E-07   .155474E-08       0    0
 9   .364575E+01  .164575E+01  -.402101E-07   .174783E-07  -.402101E-07   .174783E-07       0    1
10   .287083E+01  .870829E+00  -.415106E-07   .161154E-07  -.415106E-07   .161154E-07       0    1
```

The index numbers of the points are given under the "0-D" heading. The global coordinates of the points (x,y) given by the input are shown under the "COORDINATES" heading and the computed solution values at the point under the "SOLUTION" heading. The components of the error of the computed solution and the exact solution (if the exact solution is known and provided through the ZMPTRU subroutine) are under the "ERROR" heading. Otherwise the solution values are printed. The boundary condition numbers of the points as specified by the geometry input are under the "BDRY" heading. A zero or a one under the "EXT" heading shows whether that point is internal (0) or external (1) to the full domain.

Part of a display of the 1-D domain information is reproduced here.

```
--- 1-D INDEX=  1   B= 0, E= 1, 1/R=  .250000E+00,  FROM  1 TO  4,  PTS=   3,  R-PTS=  3
    3 POINTS=
 PT R  LOCAL        GLOBAL  COOR                     SOLUTION                      ERROR
  1 3 .500000 -.21909E+01  .33466E+01  -.185379E-07 -.931035E-08  -.185E-07 -.931E-08
        DERIV  -.672881E-01   .105613E+00  0.             0.
  2 3 .250000 -.33900E+01  .21232E+01  -.952604E-08 -.720723E-08  -.953E-08 -.721E-08
        DERIV  -.191648E+00   .118011E+00  0.             0.
  3 3 .750000 -.58997E+00  .39563E+01  -.278484E-07 -.763033E-08  -.278E-07 -.763E-08
        DERIV  -.345002E-01   .149767E+00  0.             0.
```

The index number of the 1-D domain is clearly identified. The B value is the boundary condition number of the line given by the geometry input. The E value shows whether the line is internal (0) or external (1) to the full domain. The signed value of 1/R is the curvature of that 1-D domain provided by the geometry input. The indices of the two endpoints of the 1-D domain are after "FROM" (one index) and "TO" (the other index). (The orientation of the 1-D domain is indicated by the sign of 1/R=k.) The number of points on the interior of that line is the value of "PTS". The number of regular points on that line is the value of "R-PTS".

The body of the table provides information about the points lying in the interior of that line. The indices of these points (in order of their presentation) are listed under the "PT" heading. The values of R indicate whether the point is regular (3) or not (0, 1, or 2). A regular point is a nodal point which is a corner point of all the 2-D domains on which the boundary is located. Otherwise the point is irregular. (See the User's Manual.[1]) The local coordinate of that point on the line is given under the "LOCAL" heading. The global coordinates (x,y) of that point, given under the "GLOBAL COOR" heading, are the actual coordinates of the point. The local coordinates are the coordinates of the particular point on the "master" line. The solution values of the computed solution at the point are given under the heading "SOLUTION". The components of the error between the computed solution and the exact solution (if the exact solution is known and provided through the ZPMTRU subroutine) are given under the "ERROR" heading. Otherwise the solution values are printed there.

The 2-D domain information is contained in a large display. Part of the information from that display for a sample 2-D domain is displayed here.

```
--- 2-D INDEX=  1   CORNERS=  2  1  5  4  NUMBER OF POINTS=  31  ELEMENTS=  46     ENERGY=  .14157E-07
ELT SIZES=   0   6  40   0   0   0   0   0,  SMALLER SIZES=   0,  MAX.ADJ.RATIO= 2
ERROR INDICATORS, TOTAL=  .160158E-09, MAX=  .176372E-10, MEAN=  .348169E-11, THRESHOLD=  .122084E-11
DISTR.BELOW MEAN (  35)=    2    2    2    1    7    6    8    7
DISTR.ABOVE MEAN (  11)=    2    0    1    1    1    0    2    4
DISTR.ABOVE PRED (  37)=    2    0    1    0    0    2    4   26
TIME, ASM+DEC=  .4888E+01, BCK=  .5890E+00, THR=  .1023E+01, TOTAL=  .6500uE+01
STORAGE, MEMORY=      360, AUXILIARY=      560
```

The index number of the particular 2-D domain is clearly identified as well as the index numbers of its corner points. The "NUMBER OF POINTS" is the number of regular points of the mesh for this 2-D domain. The value of "ELEMENTS" is the number of elements in the mesh for this 2-D domain. The value of "ENERGY" is the energy of the solution on this 2-D domain. The values of "ELT. SIZES" are the numbers of elements in this 2-D domain with sides of length 2^{-1}, 2^{-2}.....2^{-8}, respectively. The value of "SMALLER SIZES" is the number of elements in this 2-D domain whose side is smaller than 2^{-8}. The value of "MAX.ADJ.RATIO", the maximum adjusted ratio, is an indication of the maximum difference in sizes of two neighboring elements of this 2-D domain. The next line gives the values of the sum, the maximum, the mean, and the threshold for the error indicators of the elements in this 2-D domain. The next three lines present statistical information on the distribution of the indicators. The last two lines contain time and storage information.

Next the nodal points of this 2-D domain are listed; a few lines of this display are reproduced here.

PNT	LOCAL COORD		GLOBAL COORD		SOLUTION		ERROR	
1	.500000	.500000	-.164317E+01	.250998E+01	-.142510E-07	-.104907E-07	-.1425E-07	-.1049E-07
2	.250000	.250000	-.211873E+01	.132701E+01	-.181883E-08	-.655098E-08	-.1819E-08	-.6551E-08
3	.250000	.750000	-.296623E+01	.185782E+01	-.750303E-08	-.673864E-08	-.7503E-08	-.6739E-08
4	.750000	.250000	-.368734E+00	.247266E+01	-.224361E-07	-.128429E-07	-.2244E-07	-.1284E-07
5	.750000	.750000	-.516228E+00	.346172E+01	-.259886E-07	-.867095E-08	-.2599E-07	-.8671E-08
6	.125000	.125000	-.214080E+01	.692432E+00	.452267E-08	-.179571E-08	.4523E-08	-.1796E-08
7	.125000	.375000	-.261654E+01	.846306E+00	.520266E-09	-.212640E-08	.5203E-09	-.2126E-08
8	.375000	.125000	-.159887E+01	.158308E+01	-.449797E-08	-.107327E-07	-.4498E-08	-.1073E-07
9	.375000	.375000	-.195417E+01	.193487E+01	-.824047E-08	-.906025E-08	-.8240E-08	-.9060E-08
10	.625000	.125000	-.811366E+00	.209862E+01	-.153457E-07	-.145838E-07	-.1535E-07	-.1458E-07

The index numbers of the nodal points of this 2-D domain are given under the "PNT" heading. The nodal points are listed in order of creation. The local coordinates and the global coordinates of this nodal point are given under the headings "LOCAL COORD" and "GLOBAL COORD", respectively. The components of the computed solution for this nodal point are given under the heading "SOLUTION". The components of the error between the computed solution and the exact solution (if it is known and provided through the subroutine ZMPTRU) are given under the heading "ERROR". Otherwise the solution values are printed there. Only regular points are listed here.

Finally the elements of this 2-D domain are shown in order of decreasing error indicators. Part of one such display of elements for a given 2-D domain is given here.

ELT	H	R	LOCAL	COORD	ERROR IND.	PREV.ERR.IND.
16	3	3	.125000	.875000	.176372E-10	.254801E-09
17	3	3	.375000	.625000	.169986E-10	.254801E-09
20	3	3	.625000	.625000	.128636E-10	.266676E-09
18	3	3	.375000	.875000	.108360E-10	.254801E-09
23	3	1	.875000	.875000	.102585E-10	.266676E-09
21	3	3	.625000	.875000	.678685E-11	.266676E-09
26	4	3	.062500	.062500	.553691E-11	.283420E-09
27	4	3	.062500	.187500	.437586E-11	.283420E-09
59	4	3	.937500	.062500	.431396E-11	.277351E-09
28	4	3	.187500	.062500	.426943E-11	.283420E-09
29	4	3	.187500	.187500	.379282E-11	.283420E-09
30	4	3	.062500	.312500	.325841E-11	.283420E-09
60	4	3	.937500	.187500	.319408E-11	.277351E-09
57	4	3	.812500	.062500	.304910E-11	.277351E-09
32	4	3	.187500	.312500	.283912E-11	.283420E-09

The element indices are given under the heading "ELT". The length of the side of the element is $2^{-(h-1)}$, where h is the value of "H". The regularity condition numbers of the elements of this 2-D domain are given under the heading "R". A value of 3 means that the element has four regular corner points; values of 0, 1, or 2 mean that it does not. For more details the reader is referred to the User's Manual.[1] The local coordinates of the center of the element are given under the heading "LOCAL COORD". The present and previous error indicators for this element of this 2-D domain are given under the headings "ERROR IND." and "PREV.ERR.IND.", respectively.

(3) s-Output

The OUTPUT command prints out the components of the vector s given by Equation (2.3). Part of one such display is reproduced as Figure 4.3.

The index numbers for the 2-D domains are given under the heading "2-D". The index numbers for the elements of a specific 2-D domain are given under the heading "ELT". If h is the value of "H", then the length of the side of this element is $2^{-(h-1)}$. The components of the local and global coordinates of this element are listed under the headings "LOCAL COORD" and "GLOBAL COORD".

2-D	ELT	H	LOCAL COORD.		GLOBAL COORD.		ERR.IND.	OUTP(1)	OUTP(2)	OUTP(3)	OUTP(4)	OUTP(5)
1	16	2	.125000	.875000	-.356801E+01	.115405E+01	.1764E-10	-.917259E-02	-.823303E-01	-.267665E-01	-.396394E-08	-.306239E-08
1	17	2	.375000	.625000	-.230947E+01	.228666E+01	.1700E-10	-.430729E-01	-.458206E-01	-.618627E-01	-.108574E-07	-.830035E-08
1	18	2	.375000	.875000	-.266478E+01	.263846E+01	.1084E-10	-.401550E-01	-.406577E-01	-.470973E-01	-.130479E-07	-.821646E-08
1	20	2	.625000	.625000	-.117197E+01	.303133E+01	.1286E-10	-.883490E-01	.221324E-02	-.414717E-01	-.202795E-07	-.975812E-08
1	21	2	.625000	.875000	-.135228E+01	.349769E+01	.6787E-11	-.765082E-01	-.848083E-02	-.323636E-01	-.222499E-07	-.880531E-08
1	23	2	.875000	.875000	.369495E+00	.373175E+01	.1026E-10	-.873830E-01	.588542E-02	.112575E-01	-.317921E-07	-.456575E-08
1	26	3	.062500	.062500	-.209444E+01	.359072E+00	.5537E-11	.912400E-01	-.182197E+00	-.449752E-01	.755220E-08	.225902E-08
1	27	3	.062500	.187500	-.234085E+01	.401316E+00	.4376E-11	.647693E-01	-.152596E+00	-.357707E-01	.511412E-08	.183597E-08
1	28	3	.187500	.062500	-.192179E+01	.906837E+00	.4269E-11	.490737E-01	-.136395E+00	-.11[illegible]32E+00	.378094E-08	-.463259E-08
1	29	3	.187500	.187500	-.214788E+01	.101352E+01	.3793E-11	.308063E-01	-.116519E+00	-.877344E-01	.133379E-08	-.431848E-08
1	30	3	.062500	.312500	-.258725E+01	.443566E+00	.3258E-11	.453412E-01	-.131157E+00	-.292958E-01	.318689E-08	.134033E-08
1	31	3	.062500	.437500	-.283366E+01	.485804E+00	.2506E-11	.306292E-01	-.114058E+00	-.251048E-01	.164080E-08	.808104E-09
1	32	3	.187500	.312500	-.237397E+01	.112021E+01	.2839E-11	.175020E-01	-.102124E+00	-.711441E-01	-.636773E-09	-.422127E-08
1	33	3	.187500	.437500	-.260007E+01	.122690E+01	.2302E-11	.735261E-02	-.945209E-01	-.586405E-01	-.226470E-08	-.429611E-08
1	35	3	.312500	.062500	-.166295E+01	.132296E+01	.2111E-11	-.103002E-01	-.754348E-01	-.1413[illegible]7E+00	-.817760E-09	-.942740E-08
1	36	3	.312500	.187500	-.185859E+01	.147860E+01	.2017E-11	-.169246E-01	-.678490E-01	-.113160E+00	-.312895E-08	-.851763E-08
1	37	3	.437500	.062500	-.134347E+01	.164643E+01	.1594E-11	-.713996E-01	-.140763E-01	-.142061E+00	-.596926E-08	-.128072E-07
1	38	3	.437500	.187500	-.150152E+01	.184013E+01	.1644E-11	-.646098E-01	-.202121E-01	-.113624E+00	-.802848E-08	-.114257E-07
1	40	3	.312500	.312500	-.205424E+01	.163424E+01	.1400E-11	-.220839E-01	-.614072E-01	-.929663E-01	-.503454E-08	-.791964E-08
1	41	3	.312500	.437500	-.224988E+01	.178988E+01	.1482E-11	-.248033E-01	-.518108E-01	-.764614E-01	-.662628E-08	-.753969E-08
1	42	3	.437500	.312500	-.165957E+01	.203382E+01	.1171E-11	-.593261E-01	-.255235E-01	-.931268E-01	-.976198E-08	-.104464E-07
1	43	3	.437500	.437500	-.181763E+01	.222752E+01	.1219E-11	-.582104E-01	-.327185E-01	-.799660E-01	-.112555E-07	-.975004E-08
1	46	3	.562500	.062500	-.971590E+00	.188988E+01	.1458E-11	-.127676E+00	.424676E-01	-.117628E+00	-.115899E-07	-.149545E-07

Figure 4.3 - Components of Vector s

The present error indicator of the element is given under the heading "ERR. IND.". It should be clear that the elements are listed in the order of increasing error indicator. The components of s are defined by Equations (2.3) and (4.2). The headings "OUT(1)",...,"OUT(5)" denote σ_{xx}, σ_{yy}, σ_{xy}, u_1, u_2, respectively. These values pertain to the centers of the elements.

5. SOME TYPICAL RESULTS

Table 5.1 shows the dependence of the computed relative error (with respect to the energy norm) on the number of elements used in the long path computations.

TABLE 5.1 - RELATIVE ERROR OF THE SOLUTION

No. of Elements	Relative error
72	15.53%
150	9.27%
309	6.61%
477	4.90%

Experience shows that the ratio of the true error to the estimated error never exceeds 1.2 when the error is below 10%. Usually for an error of 5% the factor is 1.02-1.06. (In theory the ratio of the estimator to the true error approaches 1 as the error approaches zero.)

The table indicates a convergence rate of $N^{-1/2}$ as it theoretically should be.

Table 5.2 shows the number of the elements in the individual 2-D domains.

TABLE 5.2 - DISTRIBUTION OF THE ELEMENTS

Domain	No. of Elements	Domain	No. of Elements
1	46	9	28
2	64	10	19
3	49	11	52
4	61	12	49
5	4	13	16
6	19	14	4
7	19	15	13
8	34		

The distribution of the elements in the various 2-D domains is noticeably uneven.

FEARS provides a graphical display of the meshes on the master elements for every 2-D domain. Figure 5.1 shows the meshes for 2-D domain No. 8. The corners are numbered accordingly.

The computation of the finite element solution for the final mesh with 477 elements took about 60% of the total time. 5% of the final solution time was spent computing the error estimator and about 50% of this time was spent solving the linear algebraic system of equations. The time for solving the system of linear equations is roughly proportional to $N^{1.65}$ where N is the number of degrees of freedom.

The sequence of long and short paths determined by the ITERATE command was

INIT, SHORT, LONG, SHORT, LONG, SHORT, LONG

ACKNOWLEDGMENTS

The authors wish to thank Dr. Gordon C. Everstine (DTNSRDC, Code 1844) for his interest and assistance.

2-D index = 8

Number of elements = 34

Number of points = 21

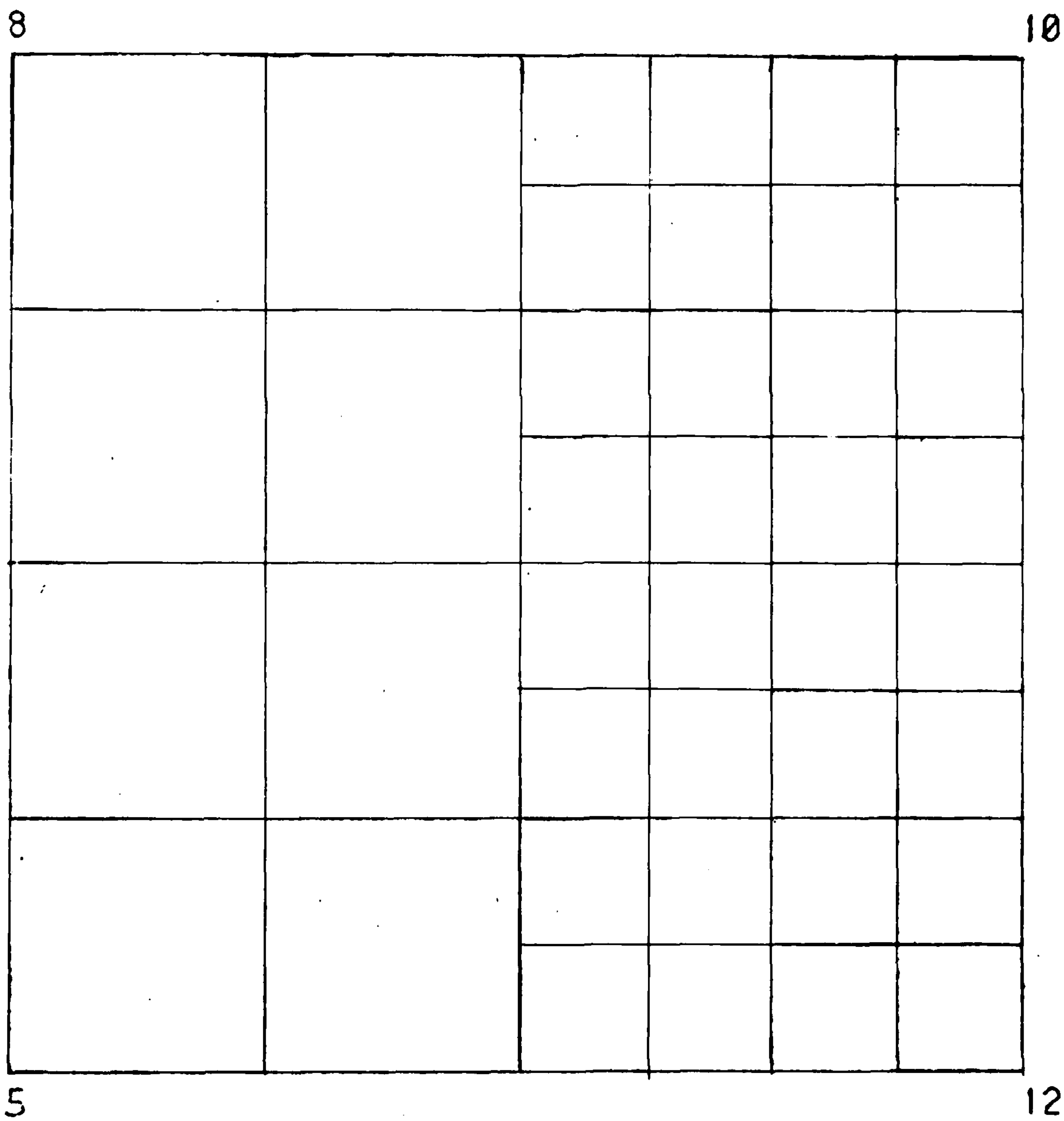

Figure 5.1 - Mesh of the 2-D Domain No. 8 on the Master Square

REFERENCE

1. Mesztenyi, C.K. "The FEARS User's Manual for the CYBER 7400 at DTNSRDC" DTNSRDC/CMLD-8 - (report to appear).

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